Algebra II

7th May 2012

Instructions. Questions carry ten marks each.

1. Let V be a finitely generated vector space over a field F. Prove that any two bases of V must have the same cardinality.

2. Let V be a vector space of dimension n over a field F. Show that the set consisting of ordered bases of V is in one-one correspondence with invertible $n \times n$ matrices with entries from F. Using this (or otherwise) find the number of $n \times n$ matrices with determinant 1 having entries from a finite field F_q of cardinality q.

3. Let W be a subspace of a vector space V. Prove that $W = V$ if and only if every basis of V contains at least one element of W.

4. Define normal matrices. Prove $\ker(A) = (\mathrm{Im}(A))^{\perp}$ for any normal matrix A.

5. For a square matrix A with complex entries, prove that $I + AA^{\star}$ is an invertible matrix.

(Hint: Prove that $AA^{\star}$ can not have -1 as an eigen value)